Happy 60th Birthday

to Costa

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KAM tori in the 1D random DNLS model and Absence of Diffusion of a wavepacket

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What is the long time behavior of an initially localized wavepacket in an infinite array of oscillators:

**Finite energy in an Infinite system**

Linear systems with **discrete linear spectrum**
(for example in disordered systems with Anderson localization)
Absence of diffusion
There is a base of square summable eigen-states (Anderson modes) on which the initial wavepacket can be expanded. The general solution is almost periodic.
No spreading.

Nonlinear systems with **purely discrete** linear spectrum.

Most common belief: Nonlinearity couples the Anderson modes and should produce « subdiffusion » (as \( t^\alpha \quad \alpha<1 \))

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Standard Kolmogorov Arnol’d Moser theory holds for finite size systems near an integrable limit

Integrable: All trajectories are quasiperiodic in time (invariant tori in the phase space)

Near Integrability (most) non resonant trajectories persist as quasiperiodic trajectories up to some critical perturbation (which depends on the considered torus)

It was believed that KAM theory does not hold for infinite systems
Almost periodic solutions (KAM tori) may exist in infinite nonlinear systems.

Exact KAM tori in nonlinear systems with discrete linear spectrum
Nondiffusive solutions

Some early rigorous results

Fröhlich Spencer Wayne (1986)
(many) Almost Periodic solutions in random Hamiltonian Systems with pair interactions
(special model)

Quasiperiodic solutions in RDNLS

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Empirical arguments

for the existence of infinite dimension tori (almost periodic solutions) with finite norm (and energy) in the 1D Random DNLS model

More generally in infinite systems with linear discrete spectrum (and exponential localization)
Anderson Representation of random DNLS

\[ i\psi_n = (\epsilon_n + |\psi_n|^2)\psi_n - C(\psi_{n+1} + \psi_{n-1}) \]

\[ \psi_n(t) = \sum_p \mu_p(t)\varphi_n^{(p)} \]

New complex variables

\[ \omega_p\varphi_n^{(p)} = \epsilon_n\varphi_n^{(p)} + C(\varphi_{n+1}^{(p)} + \varphi_{n-1}^{(p)}) \]

\[ i\mu_p = \omega_p\mu_p + \frac{\partial}{\partial \mu^*_p} \sum_n \frac{1}{2} |\psi_n|^4 = \omega_p\mu_p + \sum_{p'} C_{p,p'}\mu_{p'} \]

The Anderson modes are coupled by nonlinear terms

norm current \( p \rightarrow p' \)

\[ -J_{p\rightarrow p'} = J_{p'\rightarrow p} = 2C_{p,p'}(t)\Im(\mu^*_p\mu_{p'}) \]

\[ \approx 2\chi \sum_{q,q'} \left[ \left( \sum_n \phi_n^{(p)}\phi_n^{(p')}\phi_n^{(q)}\phi_n^{(q')} \right) |\mu_q(0)| \cdot |\mu_{q'}(0)| e^{i((\omega_q - \omega_{q'})t - (\alpha_q - \alpha_{q'}))} \right] x |\mu_p(0)| \cdot |\mu_{p'}(0)| \sin((\omega_p - \omega_{p'})t - (\alpha_p - \alpha_{p'})) \]
\[ \mu_n \sim e^{-i\omega_n t} \]
Consistency of perturbation expansion: currents integrated over long time must remain small: small denominators

Nonresonance between modes \( p \neq p' \) involving \( q, q' \):

\[ |\omega_q - \omega_{q'} \pm (\omega_p - \omega_{p'})| \gtrsim \kappa \left| \chi \sum_n \phi_n^{(p)} \phi_{n'}^{(p')} \phi_n^{(q)} \phi_{n'}^{(q')} \right| \times |\mu_q(0)| \cdot |\mu_{q'}(0)| \]
\[ \kappa \gg 1 \]

Assume random frequencies with maximum probability density \( P_0 \)

Bound for the Probability of resonance

\[ \delta_{p,p',q,q'} = 2P_0 \kappa \left| \chi \sum_n \phi_n^{(p)} \phi_{n'}^{(p')} \phi_n^{(q)} \phi_{n'}^{(q')} \right| |\mu_q(0)| \cdot |\mu_{q'}(0)| \]

If this property is fulfilled for all \( p, p', q, q' \) perturbation theory is consistent. Possibility of existence of a KAM torus

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Probability of at least one resonance

\[ P_R < \sum_{p \neq p', q, q'} \delta_{p \neq p', q, q'} \]

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\[ P_0 \kappa |\chi| \cdot |A| \cdot \|\{\mu_q(0)\}\|^2. \]

\[ P_0 \kappa |\chi| \cdot |A| \cdot \|\{\mu_q(0)\}\|^2. \]

If \( A \) has a finite norm then for

\[ \|\{\mu_q(0)\}\|^2 < \frac{1}{P_0 \kappa |\chi| \cdot |A|}, \]

\[ \|\{\mu_q(0)\}\|^2 < \frac{1}{P_0 \kappa |\chi| \cdot |A|}, \]

\[ P_R < 1 \]

The probability \( P_N = 1 - P_R \) of having no resonance is strictly positive

\[ P_N = 1 - P_R \]

\[ A \]

\( A \)

A has a finite norm when the linear spectrum is discrete and an infinite norm when it is absolutely continuous

\[ A \text{ has a finite norm when the linear spectrum is discrete and an infinite norm when it is absolutely continuous} \]

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Assumption: exponential localization

\[ |\phi_n^{(p)}| < K \sqrt{\frac{1-\lambda^2}{1+\lambda^2}} \lambda^{n-p}, \]

\[ ||A|| \leq 64K^4 \xi^2. \]

The upper bound of the norm for having no resonance (and a KAM torus?) goes to zero as the localization length diverges.

If the packet is already spread \( \mu_p \) is small and

\[ P_N > \exp \left( - P_0 \kappa |\chi| \cdot ||A|| \cdot ||\{\mu_q(0)\}||^2 \right) > 0 \]
Numerical test for the existence of KAM tori in the 1D Random DNLS model

Equivalent tests:
- Bohr recurrence
- Largest Lyapounov

Note that Poincaré recurrence theorem only holds for finite systems (and is not uniform). No recurrence is found for chaotic trajectories.
Almost periodic function: Harald Bohr theorem

Definition
\[ F(t) = \sum_n f_n e^{i\omega_n t} \text{ with } \sum_n |f_n| < \infty \]

Theorem:
This definition is equivalent to

\[ \forall \varepsilon > 0 \ \exists \{\tau_n\} \text{ (pseudo periods) such that} \]
- \[ \{\tau_n\} \text{ is relatively dense} \]
- \[ |F(t + \tau_n) - F(t)| < \varepsilon \text{ for any } t \]
Implementation for DNLS

For single site excitation at site 0, check only

$$|\Psi_0(0)|^2 - |\Psi_0(t)|^2 = N - |\Psi_0(t)|^2 < \varepsilon N$$

(because of norm conservation, no need to have recurrence in phase)

Choose $\varepsilon$ no too large for having pseudoperiods of recurrence not too large

Actually it is found that if there are recurrence at relatively large $\varepsilon$, recurrences are found at smaller $\varepsilon$ though more rarely. The pseudoperiods go roughly as $1/\varepsilon$

**Method:** Vary by small steps $\delta$ the amplitude of a single site initial condition from 0 to B. We observe for large systems and over computing time as long as possible that there are trajectories which are Bohr recurrent and others which are non recurrent.
Recurrent trajectories

- Within a given accuracy $\varepsilon$ small, many trajectories return close to their initial condition during the whole computing time (recurrent) and within bounded pseudo periods.
- Recurrence is found for smaller $\varepsilon$ but more sparsely.
- Trajectories repeat from the recurrence time, a new trajectory which is uniformly close from the initial one.
- Recurrence is observed simultaneously for all components of the trajectory.
- Some trajectories are only recurrent up to some computing time « sticking ». They become more rare as the computing time increase. Interpretation: those trajectories belongs to thin gaps of the fat Cantor set of KAM tori.
- Recurrent trajectories have zero Lyapounov exponents.
- Conversely, all trajectories found with zero Lyapounov exponents are recurrent.

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Non Recurrent trajectories

There are also many trajectories which are **not recurrent** or **loose** recurrence after some computing time,

During the computing time.

They are (or become) apparently chaotic and (start to) spread (????). Their Lyapounov exponents are non vanishing

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FIG. 1: Last observed time for recurrence in $|\psi_{n_0}|^2$ versus norm in a particular disorder realization. $\epsilon = 0.02N, W = 20, \psi_n(0) = \sqrt{N}\delta_{n,n_0}, \epsilon_{n_0} \approx 0.46529W, C = 1, \chi = -1$, system size $N = 500$. 

 Depends on disorder realisation
FIG. 2: Sets of $T$-recurrent trajectories at $T = 10^4$ for the same initial condition and disorder realization as in Fig. 1, for various disorder strengths $W$. Other parameters values are the same as in Fig. 1.
FIG. 4: Upper figure: Total norm of perturbation $\eta_n(t)$ divided by time, for solutions corresponding to single-site initial conditions $\psi_n$ with slightly different $N \approx 0.6$. Lower figure: Corresponding finite-time Lyapunov exponents. At time $10^6$, the upper curves correspond, from top to bottom, to $N = 0.604, 0.600, 0.602, 0.601$, while the lower curves for $N = 0.595, 0.599, 0.603, 0.605$ all follow very closely a curve $\sim \log t/t$, as expected for KAM tori. Disorder strength $W = 12$, other parameters and disorder realization same as in previous figures.
FIG. 3: Fraction of the trajectories $T$-recurrent at $T = 10^3$ which remain $T$-recurrent also at longer times, for various disorder strengths. Trajectories from the high-norm (self-trapped) $T$-recurrent regime have not been included. The initial condition, disorder realization, and other parameters values are the same as in Figs. 1 - 2.
Finite size studies

When the size increases, the measure of recurrent trajectories has a clearly non-vanishing asymptotic limit which is reached when the system size is sufficiently larger than the localization length (or volume).

Large fluctuations. Make disorder average for smooth graphs versus size.

If the system is periodic (no randomness), no recurrent trajectories are found as soon the system size exceeds only 10.

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Averaged for 100 realizations
Conclusions for wave packets in infinite Nonlinear systems with linear Anderson localization:
There are two kinds of wavepackets both with nonvanishing probability
- wavepackets in fat Cantor sets which are almost periodic stationary states and do not spread.
-- spreading chaotic wavepackets

Open problems
There are situations where wavepackets cannot spread totally or partially
- Do situations exist where they spread to zero.
  Blocking KAM tori. Inverse Arnold diffusion?
- If no spreading, what is the limit state?

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Many trajectories are found chaotic (with sensitivity to initial conditions) and nonvanishing Lyapounov exponent.

**Chaos does not imply wavepacket spreading.**

Self-organization?

3 examples with chaos and absence or incomplete diffusion:
- Large norm
- No linear band dispersion
- Linear system beyond a cut-off distance

Which « attractor » to expect for initially chaotic trajectories?

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Rigorous proof for the absence of complete diffusion for a « large enough » Wavepacket

\[ \| \Psi \|_2 = \left( \sum_n |\psi_n|^2 \right)^{1/2} \]
\[ \| \Psi \|_\infty = \sup_n |\psi_n| \]

with invariance by phase rotation (implies norm \( l_2 \) conservation)

\[ \psi_n \rightarrow \psi_n e^{i\theta} \]

\[ H = H_L(\{\psi_n\}) + H_{NL}(\{\psi_n\}) \]

\[ H_L = \Omega_m(\|\Psi\|_2)^2 < \langle \Psi | L | \Psi \rangle < \Omega_M(\|\Psi\|_2)^2 \]

\[ \lim_{\|\Psi\|_\infty \to 0} H_{NL}(\{\psi_n\})/(\|\Psi\|_2)^2 = 0 \]

\[ \|L\|_2 < +\infty \]

Example: \( 0 < H_{NL} = \sum_n |\psi_n|^4 < (\sup_n |\psi_n|)^2 \sum_n |\psi_n|^2 = (\|\Psi\|_\infty)^2(\|\Psi\|_2)^2 \).
Assume the wavepacket spreads uniformly to zero: \( \lim_{t=\infty} \|\Psi\|_{\infty} = 0 \), then at infinite time the nonlinear contribution \( H_{NL} \) to the energy is zero since

\[
\lim_{\|\Psi\|_{\infty} \to 0} H_{NL}(\{\psi_n\})/\|\Psi\|_2^2 = 0
\]

and the norm \( \|\Psi\|_2 \) is time constant. Then

\[
\text{At } t = +\infty, \quad H = H_L < \Omega_M(\|\Psi\|_2)^2
\]

If at \( t = 0, \quad H > \Omega_M(\|\Psi\|_2)^2 \) energy cannot be conserved and consequently the wave packet cannot spread uniformly to zero.

Since the higher order nonlinear energy grows faster than the norm the wavepacket cannot spread uniformly to zero when its amplitude is large enough.

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What could be the limit state when it is initially chaotic and when complete spreading is impossible?

A KAM torus cannot be a limit state because it is time reversible???

Self-organization?
After a sufficient long time where spreading looks chaotic, the regime of the wave packet spreading changes because it approaches KAM tori (resonances are sparse)
« inverse Arnold diffusion »

The limit state could be a « marginal » KAM torus (with weak chaos and with singular continuous spectrum )
Possible long tail in the energy density
Wave packets in infinite Nonlinear systems with linear Anderson localization

Conclusions:

- There are initial wavepackets in fat Cantor sets with FINITE PROBABILITY which are almost periodic stationary states and do not spread.

Open problems for initially chaotic wavepackets
- There are situations where they cannot spread totally or partially
- Do situations exist where they spread to zero?
- When no spreading, what is the limit state?

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