New Opportunities with Metamaterials

David R. Smith
Quantitative Metamaterial Design

Our group has specialized in the interpretation of arrays of complex scattering objects as artificial materials. 

\[
n = \frac{1}{k_d} \left\{ \pm \cos^{-1} \left[ \frac{1}{2S_{21}} \left(1 + S_{21}^2 - S_{11}^2\right) \right] + 2m\pi \right\}
\]

\[
z = \pm \frac{(1 + S_{11})^2 - S_{21}^2}{\sqrt{(1 - S_{11})^2 - S_{21}^2}}
\]
Idealized Metamaterials

\[ \varepsilon(\omega) = 1 - \frac{F \omega^2}{\omega^2 - \omega_0^2 + i\Gamma \omega} \]

\[ \mu(\omega) = 1 - \frac{F \omega^2}{\omega^2 - \omega_0^2 + i\Gamma \omega} \]

Described by relatively simple Lorentzian-like line shapes.
Real Metamaterials

Numerical retrievals, essential to metamaterial design, produce distorted lineshapes due to spatial dispersion.

Lots of subtlety in ascribing effective medium parameters to metamaterials!
Papers on Retrieval


An Analytical Metamaterial Model

Treat metamaterial as a series of infinitely thin slabs, whose $\varepsilon$ or $\mu$ go to infinity such that $\varepsilon l \to \text{const}$ or $\mu l \to \text{const}$, but $\varepsilon l^2 \to 0$.

Assume the permittivity or permeability of the slab can be described by Lorentzian forms.

Use transfer matrix, but reduce to analytical formulas.
Analytic Formulas Describe Metamaterial Line Shapes

**ELC Only**
\[
\varepsilon_B = \frac{(\alpha d / 2)}{\sin(\alpha d / 2) \cos(\alpha d / 2)}
\]
\[
\mu_B = \frac{(\alpha d / 2) \cos(\alpha d / 2)}{\sin(\alpha d / 2)}
\]
\[
\sin \frac{\alpha d}{2} = \sqrt{\varepsilon} \frac{k d}{2}
\]

**SRR Only**
\[
\varepsilon_B = \frac{(\alpha d / 2) \cos(\alpha d / 2)}{\sin(\alpha d / 2)}
\]
\[
\mu_B = \frac{\mu}{\sin(\alpha d / 2) \cos(\alpha d / 2)}
\]
\[
\sin \frac{\alpha d}{2} = \sqrt{\mu} \frac{k d}{2}
\]

Dashed lines: analytical formulas
Solid lines: full wave retrieval

R. Liu et al., PRE 76, 026606 (2007)
D. R. Smith, PRE 81, 036605 (2010)
An Analytical Metamaterial Model

If unit cells consist of two very closely spaced magnetic and electric slabs, magnetoelectric coupling occurs and the medium becomes bianisotropic.
Intracell Magnetolectric Coupling

• Coupling between elements produces significant changes in general line shapes!

• Line shape can be used to assess degree of magnetolectric coupling and bianisotropy.

\[
\sin\left(\frac{\alpha d}{2}\right) = \frac{\sqrt{\varepsilon \mu}}{2} \frac{k d}{\sqrt{1 + \varepsilon \mu \left(\frac{k d}{2}\right)^2 \kappa^2}}
\]

![Graph showing refractive index vs. frequency for different values of \(\kappa\).]
Numerical Simulations

Many magnetoelectric metamaterials can be tuned to reveal the entire range of possible coupling conditions.

Here, the spacing between the ELC and SRR in a unit cell is varied.

Dashed lines: analytical formulas
Solid lines: full wave retrieval
Progression of Optical Design: From Engineering Interfaces to Engineering Volumes

**Refractive Optics**

*Rely on refraction at an interface to control light*

**Gradient Index Optics**

*Utilize gradients in refractive index to control light*
Transformation Optics: The Next Step?

Traditional, Refractive

- Engineered interfaces
- Simple materials
- Only index considered
- Ray tracing is key tool

Transformation Optical

- Complex materials
- Geometric and wave design
- Full constitutive properties
- Reflectionless
- Coordinate transform key tool

Visualizing a coordinate transform

\[ x' = x + ay(r - b) \quad r < b \]
\[ y' = y + ax(r - b) \quad r < b \]

plot lines of constant x, constant y
Example: Compressing Space

The general prescription for transformation optical media:

\[ \varepsilon^{ij'} = \text{det} \left( \Lambda \right)^{-1} \Lambda_i^i \Lambda_j^j \varepsilon^{ij} \]
\[ \mu^{ij'} = \text{det} \left( \Lambda \right)^{-1} \Lambda_i^i \Lambda_j^j \mu^{ij} \]
\[ x'^i = \frac{\partial x'^i}{\partial x^i} x^i = \Lambda_i^i x^i \]

Transformation

\[ \frac{dx'}{dx} = \begin{cases} a, & l_1 < x < l_2 \\ 1, & x \leq l_1 \\ 1, & x \geq l_2 \end{cases} \]

Material Parameters

\[ \varepsilon^{ij} = \begin{bmatrix} a & 0 & 0 \\ 0 & 1/a & 0 \\ 0 & 0 & 1/a \end{bmatrix} n(x,y)^2 \]
\[ \mu^{ij} = \begin{bmatrix} a & 0 & 0 \\ 0 & 1/a & 0 \\ 0 & 0 & 1/a \end{bmatrix} \]
Compressing Space: Lens Flattening

To show how spatial compression works, we consider reducing the profile of a standard dielectric plano-convex lens. We apply the previous transform.

Note that the new, lower profile lens, is anisotropic. Also, part of the region that was originally free space is now an anisotropic material.
Numerical Simulation of the Flattened Lens

Original Isotropic Lens

Full Transformation Optical Lens

Eikonal Approximation of Flattened Lens
Only manage rays, not waves. Then only anisotropic index matters:

\[ n_x = \sqrt{\mu_z \varepsilon_y} \]
\[ n_y = \sqrt{\mu_z \varepsilon_x} \]
\[ \varepsilon_y' = \mu_z \varepsilon_y \]
\[ \varepsilon_x' = \mu_z \varepsilon_x \]
\[ \mu_z' = 1 \]
A Metamaterial Cloak

Design

Measurement

Simulation

NS230A Network Analyzer

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Cross Section Measurement

2D Scattering Cross Section

\[ \sigma = \lim_{R \to \infty} R \int d\theta \left| \frac{E_s(x)}{E_{inc}(x)} \right|^2 \]

Dielectric Cylinder, Test

Bare and Cloaked Metal Cylinder

- SCS reduced 24%
- SCS reduced over 2.5%

Optical Conformal Mapping

An alternative approach to TO has been suggested, based on the form invariance of the two-dimensional scalar Helmholtz equation under conformal transformations.

Conformal Transformation

\[
\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + k^2 \Psi = 0
\]

\[
4 \partial_z^* \partial_z \Psi + k^2 \Psi = 0
\]

\[
\frac{\partial^2 \Psi}{\partial u^2} + \frac{\partial^2 \Psi}{\partial v^2} + \left| \frac{dz}{dw} \right|^2 k^2 \Psi = 0
\]

Complex notation

\[
z = x + iy
\]

\[
\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial z}^*
\]

\[
w = u + iv
\]

\[
\frac{\partial}{\partial y} = i \frac{\partial}{\partial z} - i \frac{\partial}{\partial z}^*
\]

\[
w = w(z)
\]

\[
\partial_z^* \partial_z = \left| \frac{dw}{dz} \right|^2 \partial_w^* \partial_w
\]

Leonhardt et al., Science, 2006
Conformal Transform Cloak

\[ w = z + \frac{a^2}{z} \]

Transformation Grid

Index Profile

Leonhardt et al., Science, 2006

Ray trace simulations: Urzhumov et al., J. Opt., 2011
From Conformal to Quasi-Conformal

In Transformation Optics, geometry plays the major role in design, with coordinate transformations being the central tool.

Conformal transformations are interesting because they can be implemented with isotropic, non-magnetic materials.

Usually, however, exact analytic conformal transformations are not available. For these, we make use of numerical methods that yield quasi-conformal solutions.

**Conformal Cloak**
- Analytic
- Exactly conformal
- Index only
- Non-magnetic
- GO limit
- Isotropic
- Poor performance

**TO Cloak**
- Analytic
- Not conformal
- Anisotropic
- Generally magnetic
- Valid for GO
- Valid for wave optic
- Good performance
Quasi-Conformal Transformations

Consider a transformation in two dimensions:

\[ x'(x, y), \quad y'(x, y) \]

Then:

\[ \Lambda = \begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} & 0 \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

The constitutive parameters are:

\[ \epsilon = \mu = \Lambda^T \Lambda / |\Lambda| = \frac{1}{ad - bc} \begin{pmatrix} a^2 + b^2 & ac + bd & 0 \\ ac + bd & c^2 + d^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]
Numerical Transformation Generation

\[ \varepsilon = \mu = \Lambda^T \Lambda / |\Lambda| = \frac{1}{ad-bc} \begin{pmatrix} a^2 + b^2 & ac + bd & 0 \\ ac + bd & c^2 + d^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

Require: \( ac = -bd \)

\[ M \frac{\partial x'}{\partial x} = \frac{\partial y'}{\partial y}, \quad \frac{1}{M} \frac{\partial y'}{\partial x} = -\frac{\partial x'}{\partial y} \]

Cauchy-Riemann Eqs. if M=1

\[ \frac{\partial^2 x'}{\partial x^2} + \frac{\partial^2 x'}{\partial y^2} = 0, \quad \frac{\partial^2 y'}{\partial x^2} + \frac{\partial^2 y'}{\partial y^2} = 0 \]

Transformation must satisfy Laplace’s Eqs.

Quasi-conformal parameters:

\[
\begin{pmatrix}
M^2 & 0 & 0 \\
0 & M^{-2} & 0 \\
0 & 0 & \alpha 
\end{pmatrix}
\]
Ground Plane Cloak

The ground plane cloak or “carpet” cloak is an example of the utility of quasi-conformal transformations.

The ground plane cloak utilizes a numerically computed index-only medium to electromagnetically flatten a perturbed ground plane.

The transformation can be accomplished using quasi-conformal mapping, leading to index only requirement.

Li and Pendry, PRL (2008)
Optimization Using Quasi-Conformal Transformations

Note that the optimized grid exhibits much greater “squareness.”

Anisotropy for regular and QCTO grid ($\varepsilon_{xy}$)

Li and Pendry, PRL (2008)
The performance of an optimized quasi-conformal TO structure can be compared with the unoptimized case. If anisotropy and magnetic response are neglected, then performance is poor.

No material, leads to scattering from perturbation.

Ignoring $\mu$ and using isotropic $\varepsilon$ yields poor performance

Ignoring $\mu$ and using isotropic $\varepsilon$ yields improved performance
Ground-Plane Cloak by Metamaterials

R. Liu et al., Science, 2009
Dielectric Optical Carpet Cloak

- Optical Cloak
  - Low loss
  - All dielectric
  - Broadband
    - 1400 nm – 1800 nm

Xiang Zhang Group @ U.C. Berkeley

Improving Optical Devices with QCTO Transformations

- Luneburg lens is an extraordinary optical device
- Has 0 F-number
- Has infinitely wide field of view
- Perfect geometrical focus
- Has spherical focal surface

The Luneburg lens is a gradient index element, whose index has the radial profile given by:

\[ n(r) = \sqrt{2 - r^2} \]
Flattened Luneburg Design

Improving Optics

• By flattening Luneburg, we can make the lens more amenable for use with planar detectors.

• TO transformation retains all desirable properties of lens, including ultra-wide field-of-view and aberration profile.

• For 2D systems, quasi-conformal approach can be used to achieve index-only design.

• Performance can be improved via further optimization.

Kundtz et al., *Nature Materials*, 2010
Simulations and Ray Tracing

The Luneburg can be flattened using a QCTO transformation. Note the presence of regions where the index is lower than unity; these sections can be ignored without impaired performance.

n<1 regions
Flattened Luneburg Measurements

Sample Fabrication

- Index profile achieved using non-resonant metamaterials (I beams)
- Design is broadband
- n<1 regions ignored
- Prototype fabricated for microwave

Kundtz et al., Nature Materials, 2010
QCTO Transforms Can Be Broadband

Sample Fabrication

- Index profile achieved using non-resonant metamaterials (I beams)
- Design is broadband
- $n<1$ regions ignored
- Prototype fabricated for microwave

Kundtz et al., Nature Materials, 2010
A Perfect Relay Lens

- Maxwell fish eye is another example of a gradient index lens
- Radial index distribution
- Perfect imaging device to geometrical optics limit
- No geometrical aberrations
- All object/image pairs on sphere

Maxwell Lens

The Maxwell lens is a gradient index element, whose index has the radial profile given by:

\[ n(r) = \frac{n_0}{1 + \left(\frac{r}{a}\right)^2} \]

Flattened Maxwell Lens

\[ x'(x, z) = \frac{w}{a} x \]
\[ z'(x, z) = \frac{z l}{\sqrt{a^2 - x^2}} \]
Quasi-Conformal Flattened Maxwell

Transformation | QC Optimized | Full Wave Confirmation

Ray Trace Confirmation

Center for Metamaterials and Integrated Plasmonics

June 10, 2011
Experimental Results (Microwave)